Standardized Test Questions

55. True or False If $f''(c) = 0$, then $(c, f(c))$ is a point of inflection. Justify your answer.

56. True or False If $f'(c) = 0$ and $f''(c) < 0$, then $f(c)$ is a local maximum. Justify your answer.

57. Multiple Choice If $a < 0$, the graph of $y = ax^3 + 3x^2 + 4x + 5$ is concave up on

(A) $\left(-\infty, -\frac{1}{a}\right)$ (B) $\left(-\infty, \frac{1}{a}\right)$ (C) $\left(-\frac{1}{a}, \infty\right)$

(D) $\left(\frac{1}{a}, \infty\right)$ (E) $(-\infty, -1)$

58. Multiple Choice If $f(0) = f'(0) = f''(0) = 0$, which of the following *must be true*?

(A) There is a local maximum of f at the origin.

(B) There is a local minimum of f at the origin.

(C) There is no local extremum of f at the origin.

(D) There is a point of inflection of the graph of f at the origin.

(E) There is a horizontal tangent to the graph of f at the origin.

59. Multiple Choice The x-coordinates of the points of inflection of the graph of $y = x^5 - 5x^4 + 3x + 7$ are

(A) 0 only (B) 1 only (C) 3 only (D) 0 and 3 (E) 0 and 1

60. Multiple Choice Which of the following conditions would enable you to conclude that the graph of f has a point of inflection at $x = c$?

(A) There is a local maximum of f' at $x = c$.

(B) $f''(c) = 0$.

(C) $f''(c)$ does not exist.

(D) The sign of f' changes at $x = c$.

(E) f is a cubic polynomial and $c = 0$.

Exploration

61. *Graphs of Cubics* There is almost no leeway in the locations of the inflection point and the extrema of $f(x) = ax^3 + bx^2 + cx + d$, $a \neq 0$, because the one inflection point occurs at $x = -b/(3a)$ and the extrema, if any, must be located symmetrically about this value of x. Check this out by examining **(a)** the cubic in Exercise 7 and **(b)** the cubic in Exercise 2. Then **(c)** prove the general case.

Extending the Ideas

In Exercises 62 and 63, feel free to use a CAS (computer algebra system), if you have one, to solve the problem.

62. *Logistic Functions* Let $f(x) = c/(1 + ae^{-bx})$ with $a > 0$, $abc \neq 0$.

(a) Show that f is increasing on the interval $(-\infty, \infty)$ if $abc > 0$, and decreasing if $abc < 0$.

(b) Show that the point of inflection of f occurs at $x = (\ln |a|)/b$.

63. *Quartic Polynomial Functions* Let $f(x) = ax^4 + bx^3 + cx^2 + dx + e$ with $a \neq 0$.

(a) Show that the graph of f has 0 or 2 points of inflection.

(b) Write a condition that must be satisfied by the coefficients if the graph of f has 0 or 2 points of inflection.

Quick Quiz for AP* Preparation: Sections 5.1–5.3

1. Multiple Choice How many critical points does the function $f(x) = (x-2)^5(x+3)^4$ have?

(A) One (B) Two (C) Three (D) Five (E) Nine

2. Multiple Choice For what value of x does the function $f(x) = (x-2)(x-3)^2$ have a relative maximum?

(A) -3 (B) $-\frac{7}{3}$ (C) $-\frac{5}{2}$ (D) $\frac{7}{3}$ (E) $\frac{5}{2}$

3. Multiple Choice If g is a differentiable function such that $g(x) < 0$ for all real numbers x, and if $f'(x) = (x^2 - 9)g(x)$, which of the following is true?

(A) f has a relative maximum at $x = -3$ and a relative minimum at $x = 3$.

(B) f has a relative minimum at $x = -3$ and a relative maximum at $x = 3$.

(C) f has relative minima at $x = -3$ and at $x = 3$.

(D) f has relative maxima at $x = -3$ and at $x = 3$.

(E) It cannot be determined if f has any relative extrema.

4. Free Response Let f be the function given by $f(x) = 3\ln(x^2 + 2) - 2x$ with domain $[-2, 4]$.

(a) Find the coordinate of each relative maximum point and each relative minimum point of f. Justify your answer.

(b) Find the x-coordinate of each point of inflection of the graph of f.

(c) Find the absolute maximum value of $f(x)$.

5.4 Modeling and Optimization

What you will learn about . . .

- Examples from Mathematics
- Examples from Business and Industry
- Examples from Economics
- Modeling Discrete Phenomena with Differentiable Functions

and why . . .

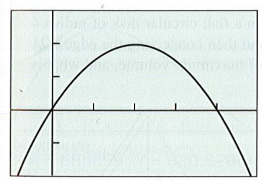
Historically, optimization problems were among the earliest applications of what we now call differential calculus.

Examples from Mathematics

While today's graphing technology makes it easy to find extrema without calculus, the algebraic methods of differentiation were understandably more practical, and certainly more accurate, when graphs had to be rendered by hand. Indeed, one of the oldest applications of what we now call "differential calculus" (pre-dating Newton and Leibniz) was to find maximum and minimum values of functions by finding where horizontal tangent lines might occur. We will use both algebraic and graphical methods in this section to solve "max-min" problems in a variety of contexts, but the emphasis will be on the *modeling* process that both methods have in common. Here is a strategy you can use:

Strategy for Solving Max-Min Problems

1. **Understand the Problem** Read the problem carefully. Identify the information you need to solve the problem.
2. **Develop a Mathematical Model of the Problem** Draw pictures and label the parts that are important to the problem. Introduce a variable to represent the quantity to be maximized or minimized. Using that variable, write a function whose extreme value gives the information sought.
3. **Graph the Function** Find the domain of the function. Determine what values of the variable make sense in the problem.
4. **Identify the Critical Points and Endpoints** Find where the derivative is zero or fails to exist.
5. **Solve the Mathematical Model** If unsure of the result, support or confirm your solution with another method.
6. **Interpret the Solution** Translate your mathematical result into the problem setting and decide whether the result makes sense.



[–5, 25] by [–100, 150]

Figure 5.35 The graph of $f(x) = x(20 - x)$ with domain $(-\infty, \infty)$ has an absolute maximum of 100 at $x = 10$. (Example 1)

EXAMPLE 1 Using the Strategy

Find two numbers whose sum is 20 and whose product is as large as possible.

SOLUTION

Model If one number is x, the other is $(20 - x)$, and their product is $f(x) = x(20 - x)$.

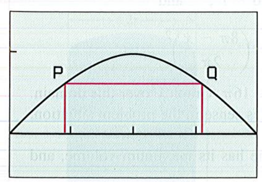
Solve Graphically We can see from the graph of f in Figure 5.35 that there is a maximum. From what we know about parabolas, the maximum occurs at $x = 10$.

Confirm Analytically When $x = 10$, $f'(x) = 20 - 2x = 0$. Since $f''(x) = -2$ (always negative), the maximum occurs at $x = 10$. The other number is $20 - x = 10$.

Interpret The two numbers we seek are $x = 10$ and $20 - x = 10$.

Now Try Exercise 1.

Sometimes we find it helpful to use both analytic and graphical methods together, as in Example 2.



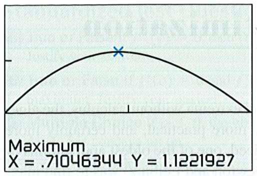
[0, π] by [–0.5, 1.5]

Figure 5.36 A rectangle inscribed under one arch of $y = \sin x$. (Example 2)

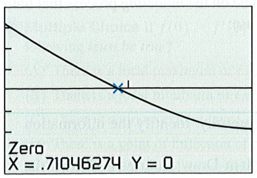
EXAMPLE 2 Inscribing Rectangles

A rectangle is to be inscribed under one arch of the sine curve (Figure 5.36). What is the largest area the rectangle can have, and what dimensions give that area?

continued



[0, $\pi/2$] by [–1, 2]
(a)



[0, $\pi/2$] by [–4, 4]
(b)

Figure 5.37 The graph of (a) $A(x) = (\pi - 2x)\sin x$ and (b) A' in the interval $0 \le x \le \pi/2$. (Example 2)

SOLUTION

Model Let $(x, \sin x)$ be the coordinates of point P in Figure 5.36. From what we know about the sine function the x-coordinate of point Q is $(\pi - x)$. Thus,

$$\pi - 2x = \text{length of rectangle}$$

and

$$\sin x = \text{height of rectangle.}$$

The area of the rectangle is

$$A(x) = (\pi - 2x)\sin x.$$

Solve Analytically and Graphically We can assume that $0 \le x \le \pi/2$. Notice that $A = 0$ at the endpoints $x = 0$ and $x = \pi/2$. Since A is differentiable, the only critical points occur at the zeros of the first derivative,

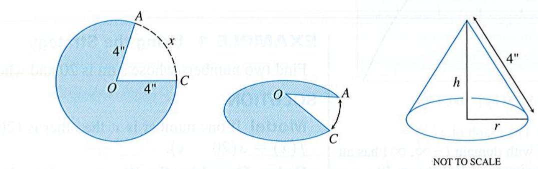
$$A'(x) = -2\sin x + (\pi - 2x)\cos x.$$

It is not possible to solve the equation $A'(x) = 0$ using algebraic methods. We can use the graph of A (Figure 5.37a) to find the maximum value and where it occurs. Or, we can use the graph of A' (Figure 5.37b) to find where the derivative is zero, and then evaluate A at this value of x to find the maximum value. The two x-values appear to be the same, as they should.

Interpret The rectangle has a maximum area of about 1.122 square units when $x \approx 0.710$. At this point, the rectangle is $\pi - 2x \approx 1.721$ units long by $\sin x \approx 0.652$ unit high. *Now Try Exercise 5.*

EXPLORATION 1 Constructing Cones

A cone of height h and radius r is constructed from a flat, circular disk of radius 4 in. by removing a sector AOC of arc length x in. and then connecting the edges OA and OC. What arc length x will produce the cone of maximum volume, and what is that volume?

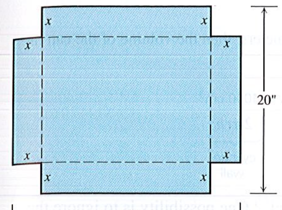


1. Show that

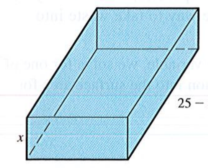
$$r = \frac{8\pi - x}{2\pi}, \quad h = \sqrt{16 - r^2}, \quad \text{and}$$

$$V(x) = \frac{\pi}{3}\left(\frac{8\pi - x}{2\pi}\right)^2 \sqrt{16 - \left(\frac{8\pi - x}{2\pi}\right)^2}$$

2. Show that the natural domain of V is $0 \le x \le 16\pi$. Graph V over this domain.
3. Explain why the restriction $0 \le x \le 8\pi$ makes sense in the problem situation. Graph V over this domain.
4. Use graphical methods to find where the cone has its maximum volume, and what that volume is.
5. Confirm your findings in part 4 analytically. [*Hint:* Use $V(x) = (1/3)\pi r^2 h$, $h^2 + r^2 = 16$, and the Chain Rule.]



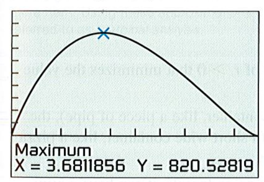
(a)



(b)

Figure 5.38 An open box made by cutting the corners from a piece of tin. (Example 3)

$y = x(20 - 2x)(25 - 2x)$



[0, 10] by [–300, 1000]

Figure 5.39 We chose the –300 in $-300 \le y \le 1000$ so that the coordinates of the local maximum at the bottom of the screen would not interfere with the graph. (Example 3)

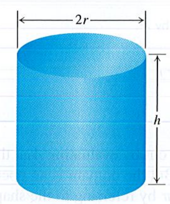


Figure 5.40 This one-liter can uses the least material when $h = 2r$. (Example 4)

Examples from Business and Industry

To *optimize* something means to maximize or minimize some aspect of it. What is the size of the most profitable production run? What is the least expensive shape for an oil can? What is the stiffest rectangular beam we can cut from a 12-inch log? We usually answer such questions by finding the greatest or smallest value of some function that we have used to model the situation.

EXAMPLE 3 Fabricating a Box

An open-top box is to be made by cutting congruent squares of side length x from the corners of a 20- by 25-inch sheet of tin and bending up the sides (Figure 5.38). How large should the squares be to make the box hold as much as possible? What is the resulting maximum volume?

SOLUTION

Model The height of the box is x, and the other two dimensions are $(20 - 2x)$ and $(25 - 2x)$. Thus, the volume of the box is

$$V(x) = x(20 - 2x)(25 - 2x).$$

Solve Analytically Expanding, we obtain $V(x) = 4x^3 - 90x^2 + 500x$. The first derivative of V is

$$V'(x) = 12x^2 - 180x + 500.$$

The two solutions of the quadratic equation $V'(x) = 0$ are

$$c_1 = \frac{180 - \sqrt{180^2 - 48(500)}}{24} \approx 3.681 \quad \text{and}$$

$$c_2 = \frac{180 + \sqrt{180^2 - 48(500)}}{24} \approx 11.317.$$

Only c_1 is in the domain $[0, 10]$ of V. The values of V at this one critical point and the two endpoints are

Critical point value: $V(c_1) \approx 820.528$
Endpoint values: $V(0) = 0, \quad V(10) = 0.$

Support Graphically Because $2x$ cannot exceed 20, we have $0 \le x \le 10$. Figure 5.39 suggests that the maximum value of V is about 820.528 and occurs at $x \approx 3.681$.

Interpret Cutout squares that are about 3.681 in. on a side give the maximum volume, about 820.528 in^3. *Now Try Exercise 7.*

EXAMPLE 4 Designing a Can

You have been asked to design a one-liter oil can shaped like a right circular cylinder (see Figure 5.40). What dimensions will use the least material?

continued

SOLUTION

Volume of can: If r and h are measured in centimeters, then the volume of the can in cubic centimeters is

$$\pi r^2 h = 1000. \quad \text{1 liter} = 1000 \text{ cm}^3$$

$$\textit{Surface area of can:} \quad A = \underbrace{2\pi r^2}_{\substack{\text{circular}\\ \text{ends}}} + \underbrace{2\pi r h}_{\substack{\text{cylinder}\\ \text{wall}}}$$

How can we interpret the phrase "least material"? One possibility is to ignore the thickness of the material and the waste in manufacturing. Then we ask for dimensions r and h that make the total surface area as small as possible while satisfying the constraint $\pi r^2 h = 1000$. (Exercise 17 describes one way to take waste into account.)

Model To express the surface area as a function of one variable, we solve for one of the variables in $\pi r^2 h = 1000$ and substitute that expression into the surface area formula. Solving for h is easier,

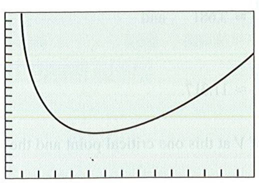
$$h = \frac{1000}{\pi r^2}.$$

Thus,

$$A = 2\pi r^2 + 2\pi r h$$

$$= 2\pi r^2 + 2\pi r\left(\frac{1000}{\pi r^2}\right)$$

$$= 2\pi r^2 + \frac{2000}{r}.$$



[0, 15] by [0, 2000]

Figure 5.41 The graph of $A = 2\pi r^2 + 2000/r$, $r > 0$, suggests that the minimum occurs when the radius is about 5.419 cm. (Example 4)

Solve Analytically Our goal is to find a value of $r > 0$ that minimizes the value of A. Figure 5.41 suggests that such a value exists.

Notice from the graph that for small r (a tall thin container, like a piece of pipe), the term $2000/r$ dominates and A is large. For large r (a short wide container, like a pizza pan), the term $2\pi r^2$ dominates and A again is large.

Since A is differentiable on $r > 0$, an interval with no endpoints, it can have a minimum value only where its first derivative is zero.

$$\frac{dA}{dr} = 4\pi r - \frac{2000}{r^2}$$

$$0 = 4\pi r - \frac{2000}{r^2} \quad \text{Set } dA/dr = 0.$$

$$4\pi r^3 = 2000 \quad \text{Multiply by } r^2.$$

$$r = \sqrt[3]{\frac{500}{\pi}} \approx 5.419 \quad \text{Solve for } r.$$

Something happens at $r = \sqrt[3]{500/\pi}$, but what?

If the domain of A were a closed interval, we could find out by evaluating A at this critical point and the endpoints and comparing the results. But the domain is an open interval, so we must learn what is happening at $r = \sqrt[3]{500/\pi}$ by referring to the shape of A's graph. The second derivative

continued

$$\frac{d^2A}{dr^2} = 4\pi + \frac{4000}{r^3}$$

is positive throughout the domain of A. The graph is therefore concave up and the value of A at $r = \sqrt[3]{500/\pi}$ an absolute minimum.

The corresponding value of h (after a little algebra) is

$$h = \frac{1000}{\pi r^2} = 2\sqrt[3]{\frac{500}{\pi}} = 2r.$$

Interpret The one-liter can that uses the least material has height equal to the diameter, with $r \approx 5.419$ cm and $h \approx 10.839$ cm. ***Now Try Exercise 11.***

Examples from Economics

Here we want to point out two more places where calculus makes a contribution to economic theory. The first has to do with maximizing profit. The second has to do with minimizing average cost.

Suppose that

$$r(x) = \text{the revenue from selling } x \text{ items,}$$
$$c(x) = \text{the cost of producing the } x \text{ items,}$$
$$p(x) = r(x) - c(x) = \text{the profit from selling } x \text{ items.}$$

Marginal Analysis

Because differentiable functions are locally linear, we can use the marginals to approximate the extra revenue, cost, or profit resulting from selling or producing one more item. Using these approximations is referred to as *marginal analysis*.

The marginal revenue, marginal cost, and marginal profit at this production level (x items) are

$$\frac{dr}{dx} = \text{marginal revenue}, \quad \frac{dc}{dx} = \text{marginal cost}, \quad \frac{dp}{dx} = \text{marginal profit}.$$

The first observation is about the relationship of p to these derivatives.

THEOREM 6 Maximum Profit

Maximum profit (if any) occurs at a production level at which marginal revenue equals marginal cost.

Proof We assume that $r(x)$ and $c(x)$ are differentiable for all $x > 0$, so if $p(x) = r(x) - c(x)$ has a maximum value, it occurs at a production level at which $p'(x) = 0$. Since $p'(x) = r'(x) - c'(x)$, $p'(x) = 0$ implies that

$$r'(x) - c'(x) = 0 \quad \text{or} \quad r'(x) = c'(x). \quad ■$$

Figure 5.42 gives more information about this situation.

What guidance do we get from this observation? We know that a production level at which $p'(x) = 0$ need not be a level of maximum profit. It might be a level of minimum

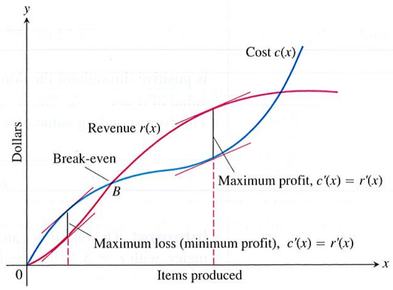


Figure 5.42 The graph of a typical cost function starts concave down and later turns concave up. It crosses the revenue curve at the break-even point B. To the left of B, the company operates at a loss. To the right, the company operates at a profit, the maximum profit occurring where $r'(x) = c'(x)$. Farther to the right, cost exceeds revenue (perhaps because of a combination of market saturation and rising labor and material costs) and production levels become unprofitable again.

profit, for example. But if we are making financial projections for our company, we should look for production levels at which marginal cost seems to equal marginal revenue. If there is a most profitable production level, it will be one of these.

EXAMPLE 5 Maximizing Profit

Suppose that $r(x) = 9x$ and $c(x) = x^3 - 6x^2 + 15x$, where x represents thousands of units. Is there a production level that maximizes profit? If so, what is it?

SOLUTION

Notice that $r'(x) = 9$ and $c'(x) = 3x^2 - 12x + 15$.

$$3x^2 - 12x + 15 = 9 \quad \text{Set } c'(x) = r'(x).$$
$$3x^2 - 12x + 6 = 0$$

The two solutions of the quadratic equation are

$$x_1 = \frac{12 - \sqrt{72}}{6} = 2 - \sqrt{2} \approx 0.586 \quad \text{and}$$

$$x_2 = \frac{12 + \sqrt{72}}{6} = 2 + \sqrt{2} \approx 3.414.$$

The possible production levels for maximum profit are $x \approx 0.586$ thousand units or $x \approx 3.414$ thousand units. The graphs in Figure 5.43 show that maximum profit occurs at about $x = 3.414$ and maximum loss occurs at about $x = 0.586$.

Another way to look for optimal production levels is to look for levels that minimize the average cost of the units produced. Theorem 7 helps us find them. ***Now Try Exercise 23.***

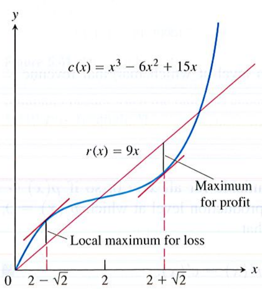


Figure 5.43 The cost and revenue curves for Example 5.

THEOREM 7 Minimizing Average Cost

The production level (if any) at which average cost is smallest is a level at which the average cost equals the marginal cost.

Proof We assume that $c(x)$ is differentiable.

$$c(x) = \text{cost of producing } x \text{ items, } x > 0.$$

$$\frac{c(x)}{x} = \text{average cost of producing } x \text{ items}$$

If the average cost can be minimized, it will be a production level at which

$$\frac{d}{dx}\left(\frac{c(x)}{x}\right) = 0$$

$$\frac{xc'(x) - c(x)}{x^2} = 0 \quad \text{Quotient Rule}$$

$$xc'(x) - c(x) = 0 \quad \text{Multiply by } x^2.$$

$$\underbrace{c'(x)}_{\text{marginal cost}} = \underbrace{\frac{c(x)}{x}}_{\text{average cost}}.$$

■

Again we have to be careful about what Theorem 7 does and does not say. It does not say that there is a production level of minimum average cost—it says where to look to see if there is one. Look for production levels at which average cost and marginal cost are equal. Then check to see if any of them gives a minimum average cost.

EXAMPLE 6 Minimizing Average Cost

Suppose $c(x) = x^3 - 6x^2 + 15x$, where x represents thousands of units. Is there a production level that minimizes average cost? If so, what is it?

SOLUTION

We look for levels at which average cost equals marginal cost.

$$\text{Marginal cost: } c'(x) = 3x^2 - 12x + 15$$

$$\text{Average cost: } \frac{c(x)}{x} = x^2 - 6x + 15$$

$$3x^2 - 12x + 15 = x^2 - 6x + 15 \quad \text{Marginal cost = Average cost}$$
$$2x^2 - 6x = 0$$
$$2x(x - 3) = 0$$
$$x = 0 \quad \text{or} \quad x = 3$$

Since $x > 0$, the only production level that might minimize average cost is $x = 3$ thousand units.

We use the Second Derivative Test.

$$\frac{d}{dx}\left(\frac{c(x)}{x}\right) = 2x - 6$$

$$\frac{d^2}{dx^2}\left(\frac{c(x)}{x}\right) = 2 > 0$$

The second derivative is positive for all $x > 0$, so $x = 3$ gives an absolute minimum. ***Now Try Exercise 25.***

Modeling Discrete Phenomena with Differentiable Functions

In case you are wondering how we can use differentiable functions $c(x)$ and $r(x)$ to describe the cost and revenue that comes from producing a number of items x that can only be an integer, here is the rationale.

When x is large, we can reasonably fit the cost and revenue data with smooth curves $c(x)$ and $r(x)$ that are defined not only at integer values of x but also at the values in between just as we do when we use regression equations. Once we have these differentiable functions, which are supposed to behave like the real cost and revenue when x is an integer, we can apply calculus to draw conclusions about their values. We then translate these mathematical conclusions into inferences about the real world that we hope will have predictive value. When they do, as is the case with the economic theory here, we say that the functions give a good model of reality.

What do we do when our calculus tells us that the best production level is a value of x that isn't an integer, as it did in Example 5? We use the nearest convenient integer. For $x \approx 3.414$ thousand units in Example 5, we might use 3414, or perhaps 3410 or 3420 if we ship in boxes of 10.

Quick Review 5.4 *(For help, go to Sections 1.6, 5.1, and Appendix A.1.)*

Exercise numbers with a gray background indicate problems that the authors have designed to be solved *without a calculator.*

1. Use the First Derivative Test to identify the local extrema of $y = x^3 - 6x^2 + 12x - 8$.

2. Use the Second Derivative Test to identify the local extrema of $y = 2x^3 + 3x^2 - 12x - 3$.

3. Find the volume of a cone with radius 5 cm and height 8 cm.

4. Find the dimensions of a right circular cylinder with volume 1000 cm^3 and surface area 600 cm^2.

In Exercises 5–8, rewrite the expression as a trigonometric function of the angle α.

5. $\sin(-\alpha)$ **6.** $\cos(-\alpha)$

7. $\sin(\pi - \alpha)$ **8.** $\cos(\pi - \alpha)$

In Exercises 9 and 10, use substitution to find the exact solutions of the system of equations.

9. $\begin{cases} x^2 + y^2 = 4 \\ y = \sqrt{3x} \end{cases}$

10. $\begin{cases} \dfrac{x^2}{4} + \dfrac{y^2}{9} = 1 \\ y = x + 3 \end{cases}$

Section 5.4 Exercises

In Exercises 1–10, solve the problem analytically. Support your answer graphically.

1. *Finding Numbers* The sum of two nonnegative numbers is 20. Find the numbers if

(a) the sum of their squares is as large as possible; as small as possible.

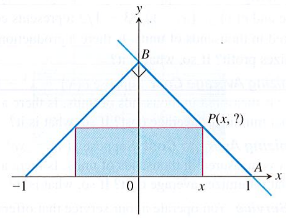
(b) one number plus the square root of the other is as large as possible; as small as possible.

2. *Maximizing Area* What is the largest possible area for a right triangle whose hypotenuse is 5 cm long, and what are its dimensions?

3. *Maximizing Perimeter* What is the smallest perimeter possible for a rectangle whose area is 16 in^2, and what are its dimensions?

4. *Finding Area* Show that among all rectangles with an 8-m perimeter, the one with largest area is a square.

5. *Inscribing Rectangles* The figure shows a rectangle inscribed in an isosceles right triangle whose hypotenuse is 2 units long.



(a) Express the y-coordinate of P in terms of x. [*Hint:* Write an equation for the line AB.]

(b) Express the area of the rectangle in terms of x.

(c) What is the largest area the rectangle can have, and what are its dimensions?

6. *Largest Rectangle* A rectangle has its base on the x-axis and its upper two vertices on the parabola $y = 12 - x^2$. What is the largest area the rectangle can have, and what are its dimensions?

7. *Optimal Dimensions* You are planning to make an open rectangular box from an 8- by 15-in. piece of cardboard by cutting congruent squares from the corners and folding up the sides. What are the dimensions of the box of largest volume you can make this way, and what is its volume?

8. *Closing Off the First Quadrant* You are planning to close off a corner of the first quadrant with a line segment 20 units long running from $(a, 0)$ to $(0, b)$. Show that the area of the triangle enclosed by the segment is largest when $a = b$.

9. *The Best Fencing Plan* A rectangular plot of farmland will be bounded on one side by a river and on the other three sides by a single-strand electric fence. With 800 m of wire at your disposal, what is the largest area you can enclose, and what are its dimensions?

10. *The Shortest Fence* A 216-m^2 rectangular pea patch is to be enclosed by a fence and divided into two equal parts by another fence parallel to one of the sides. What dimensions for the outer rectangle will require the smallest total length of fence? How much fence will be needed?

11. *Designing a Tank* Your iron works has contracted to design and build a 500-ft^3, square-based, open-top, rectangular steel holding tank for a paper company. The tank is to be made by welding thin stainless steel plates together along their edges. As the production engineer, your job is to find dimensions for the base and height that will make the tank weigh as little as possible.

(a) What dimensions do you tell the shop to use?

(b) **Writing to Learn** Briefly describe how you took weight into account.

12. *Catching Rainwater* A 1125-ft^3 open-top rectangular tank with a square base x ft on a side and y ft deep is to be built with its top flush with the ground to catch runoff water. The costs associated with the tank involve not only the material from which the tank is made but also an excavation charge proportional to the product xy.

(a) If the total cost is

$$c = 5(x^2 + 4xy) + 10xy,$$

what values of x and y will minimize it?

(b) **Writing to Learn** Give a possible scenario for the cost function in (a).

13. *Designing a Poster* You are designing a rectangular poster to contain 50 in^2 of printing with a 4-in. margin at the top and bottom and a 2-in. margin at each side. What overall dimensions will minimize the amount of paper used?

14. *Vertical Motion* The height of an object moving vertically is given by

$$s = -16t^2 + 96t + 112,$$

with s in ft and t in sec. Find **(a)** the object's velocity when $t = 0$, **(b)** its maximum height and when it occurs, and **(c)** its velocity when $s = 0$.

15. *Finding an Angle* Two sides of a triangle have lengths a and b, and the angle between them is θ. What value of θ will maximize the triangle's area? [*Hint:* $A = (1/2)\, ab \sin\theta$.]

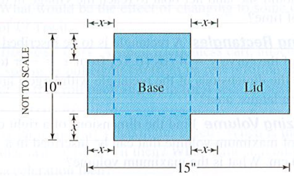
16. *Designing a Can* What are the dimensions of the lightest open-top right circular cylindrical can that will hold a volume of 1000 cm^3? Compare the result here with the result in Example 4.

17. *Designing a Can* You are designing a 1000-cm^3 right circular cylindrical can whose manufacture will take waste into account. There is no waste in cutting the aluminum for the side, but the top and bottom of radius r will be cut from squares that measure $2r$ units on a side. The total amount of aluminum used up by the can will therefore be

$$A = 8r^2 + 2\pi rh$$

rather than the $A = 2\pi r^2 + 2\pi rh$ in Example 4. In Example 4 the ratio of h to r for the most economical can was 2 to 1. What is the ratio now?

18. *Designing a Box with Lid* A piece of cardboard measures 10 in. by 15 in. Two equal squares are removed from the corners of a 10-in. side as shown in the figure. Two equal rectangles are removed from the other corners so that the tabs can be folded to form a rectangular box with lid.



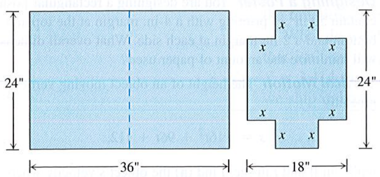
(a) Write a formula $V(x)$ for the volume of the box.

(b) Find the domain of V for the problem situation and graph V over this domain.

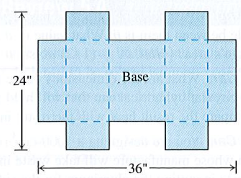
(c) Use a graphical method to find the maximum volume and the value of x that gives it.

(d) Confirm your result in part (c) analytically.

19. ***Designing a Suitcase*** A 24- by 36-in. sheet of cardboard is folded in half to form a 24- by 18-in. rectangle as shown in the figure. Then four congruent squares of side length x are cut from the corners of the folded rectangle. The sheet is unfolded, and the six tabs are folded up to form a box with sides and a lid.



The sheet is then unfolded.



(a) Write a formula $V(x)$ for the volume of the box.

(b) Find the domain of V for the problem situation and graph V over this domain.

(c) Use an analytic method to find the maximum volume and the value of x that gives it.

(d) Support your result in part (c) graphically.

(e) Find a value of x that yields a volume of 1120 in^3.

(f) **Writing to Learn** Write a paragraph describing the issues that arise in part (b).

20. ***Quickest Route*** Jane is 2 mi offshore in a boat and wishes to reach a coastal village 6 mi down a straight shoreline from the point nearest the boat. She can row 2 mph and can walk 5 mph. Where should she land her boat to reach the village in the least amount of time?

21. ***Inscribing Rectangles*** A rectangle is to be inscribed under the arch of the curve $y = 4\cos(0.5x)$ from $x = -\pi$ to $x = \pi$. What are the dimensions of the rectangle with largest area, and what is the largest area?

22. ***Maximizing Volume*** Find the dimensions of a right circular cylinder of maximum volume that can be inscribed in a sphere of radius 10 cm. What is the maximum volume?

23. ***Maximizing Profit*** Suppose $r(x) = 8\sqrt{x}$ represents revenue and $c(x) = 2x^2$ represents cost, with x measured in thousands of units. Is there a production level that maximizes profit? If so, what is it?

24. ***Maximizing Profit*** Suppose $r(x) = x^2/(x^2 + 1)$ represents revenue and $c(x) = (x - 1)^3/3 - 1/3$ represents cost, with x measured in thousands of units. Is there a production level that maximizes profit? If so, what is it?

25. ***Minimizing Average Cost*** Suppose $c(x) = x^3 - 10x^2 - 30x$, where x is measured in thousands of units. Is there a production level that minimizes average cost? If so, what is it?

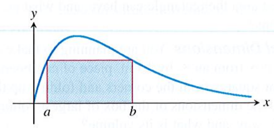
26. ***Minimizing Average Cost*** Suppose $c(x) = xe^x - 2x^2$, where x is measured in thousands of units. Is there a production level that minimizes average cost? If so, what is it?

27. ***Tour Service*** You operate a tour service that offers the following rates:

- \$200 per person if 50 people (the minimum number to book the tour) go on the tour.
- For each additional person, up to a maximum of 80 people total, the rate per person is reduced by \$2.

It costs \$6000 (a fixed cost) plus \$32 per person to conduct the tour. How many people does it take to maximize your profit?

28. **Group Activity** The figure shows the graph of $f(x) = xe^{-x}$, $x \geq 0$.



(a) Find where the absolute maximum of f occurs.

(b) Let $a > 0$ and $b > 0$ be given as shown in the figure. Complete the following table where A is the area of the rectangle in the figure.

a	b	A
0.1		
0.2		
0.3		
⋮		
1		

(c) Draw a scatter plot of the data (a, A).

(d) Find the quadratic, cubic, and quartic regression equations for the data in part (b), and superimpose their graphs on a scatter plot of the data.

(e) Use each of the regression equations in part (d) to estimate the maximum possible value of the area of the rectangle.

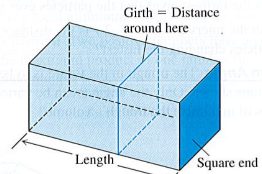
29. ***Cubic Polynomial Functions***

Let $f(x) = ax^3 + bx^2 + cx + d$, $a \neq 0$.

(a) Show that f has either 0 or 2 local extrema.

(b) Give an example of each possibility in part (a).

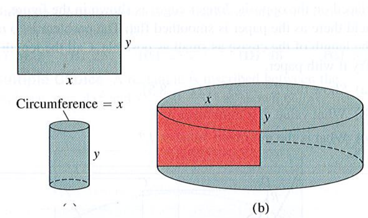
30. ***Shipping Packages*** The U.S. Postal Service will accept a box for domestic shipment only if the sum of its length and girth (distance around), as shown in the figure, does not exceed 108 in. What dimensions will give a box with a square end the largest possible volume?



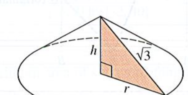
31. ***Constructing Cylinders*** Compare the answers to the following two construction problems.

(a) A rectangular sheet of perimeter 36 cm and dimensions x cm by y cm is to be rolled into a cylinder as shown in part (a) of the figure. What values of x and y give the largest volume?

(b) The same sheet is to be revolved about one of the sides of length y to sweep out the cylinder as shown in part (b) of the figure. What values of x and y give the largest volume?



32. ***Constructing Cones*** A right triangle whose hypotenuse is $\sqrt{3}$ m long is revolved about one of its legs to generate a right circular cone. Find the radius, height, and volume of the cone of greatest volume that can be made this way.

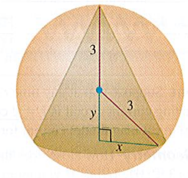


33. ***Finding Parameter Values*** What value of a makes $f(x) = x^2 + (a/x)$ have **(a)** a local minimum at $x = 2$? **(b)** a point of inflection at $x = 1$?

34. ***Finding Parameter Values*** Show that $f(x) = x^2 + (a/x)$ cannot have a local maximum for any value of a.

35. ***Finding Parameter Values*** What values of a and b make $f(x) = x^3 + ax^2 + bx$ have **(a)** a local maximum at $x = -1$ and a local minimum at $x = 3$? **(b)** a local minimum at $x = 4$ and a point of inflection at $x = 1$?

36. ***Inscribing a Cone*** Find the volume of the largest right circular cone that can be inscribed in a sphere of radius 3.

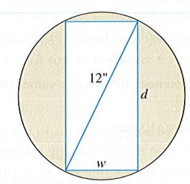


37. ***Strength of a Beam*** The strength S of a rectangular wooden beam is proportional to its width times the square of its depth.

(a) Find the dimensions of the strongest beam that can be cut from a 12-in.-diameter cylindrical log.

(b) **Writing to Learn** Graph S as a function of the beam's width w, assuming the proportionality constant to be $k = 1$. Reconcile what you see with your answer in part (a).

(c) **Writing to Learn** On the same screen, graph S as a function of the beam's depth d, again taking $k = 1$. Compare the graphs with one another and with your answer in part (a). What would be the effect of changing to some other value of k? Try it.



38. ***Stiffness of a Beam*** The stiffness S of a rectangular beam is proportional to its width times the cube of its depth.

(a) Find the dimensions of the stiffest beam that can be cut from a 12-in.-diameter cylindrical log.

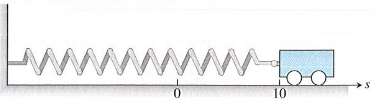
(b) **Writing to Learn** Graph S as a function of the beam's width w, assuming the proportionality constant to be $k = 1$. Reconcile what you see with your answer in part (a).

(c) **Writing to Learn** On the same screen, graph S as a function of the beam's depth d, again taking $k = 1$. Compare the graphs with one another and with your answer in part (a). What would be the effect of changing to some other value of k? Try it.

39. ***Frictionless Cart*** A small frictionless cart, attached to the wall by a spring, is pulled 10 cm from its rest position and released at time $t = 0$ to roll back and forth for 4 sec. Its position at time t is $s = 10\cos\pi t$.

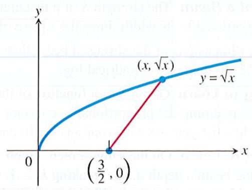
(a) What is the cart's maximum speed? When is the cart moving that fast? Where is it then? What is the magnitude of the acceleration then?

(b) Where is the cart when the magnitude of the acceleration is greatest? What is the cart's speed then?



40. *Electrical Current* Suppose that at any time t (sec) the current i (amp) in an alternating current circuit is $i = 2\cos t + 2\sin t$. What is the peak (largest magnitude) current for this circuit?

41. *Calculus and Geometry* How close does the curve $y = \sqrt{x}$ come to the point $(3/2, 0)$? [*Hint:* If you minimize the *square* of the distance, you can avoid square roots.]



42. *Calculus and Geometry* How close does the semicircle $y = \sqrt{16 - x^2}$ come to the point $(1, \sqrt{3})$?

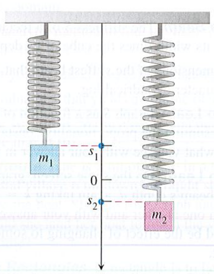
43. Writing to Learn Is the function $f(x) = x^2 - x + 1$ ever negative? Explain.

44. Writing to Learn You have been asked to determine whether the function $f(x) = 3 + 4\cos x + \cos 2x$ is ever negative.

(a) Explain why you need to consider values of x only in the interval $[0, 2\pi]$.

(b) Is f ever negative? Explain.

45. *Vertical Motion* Two masses hanging side by side from springs have positions $s_1 = 2\sin t$ and $s_2 = \sin 2t$, respectively, with s_1 and s_2 in meters and t in seconds.



(a) At what times in the interval $t > 0$ do the masses pass each other? [*Hint:* $\sin 2t = 2\sin t\cos t$.]

(b) When in the interval $0 \le t \le 2\pi$ is the vertical distance between the masses the greatest? What is this distance? [*Hint:* $\cos 2t = 2\cos^2 t - 1$.]

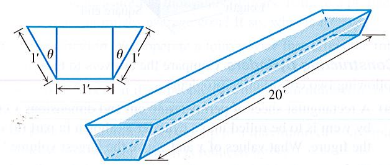
46. *Motion on a Line* The positions of two particles on the s-axis are $s_1 = \sin t$ and $s_2 = \sin(t + \pi/3)$, with s_1 and s_2 in meters and t in seconds.

(a) At what time(s) in the interval $0 \le t \le 2\pi$ do the particles meet?

(b) What is the farthest apart that the particles ever get?

(c) When in the interval $0 \le t \le 2\pi$ is the distance between the particles changing the fastest?

47. *Finding an Angle* The trough in the figure is to be made to the dimensions shown. Only the angle θ can be varied. What value of θ will maximize the trough's volume?

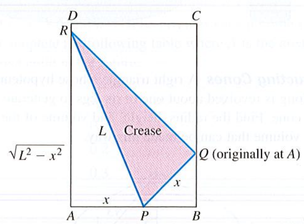


48. Group Activity *Paper Folding* A rectangular sheet of 8 1/2-by 11-in. paper is placed on a flat surface. One of the corners is placed on the opposite longer edge, as shown in the figure, and held there as the paper is smoothed flat. The problem is to make the length of the crease as small as possible. Call the length L. Try it with paper.

(a) Show that $L^2 = 2x^3/(2x - 8.5)$.

(b) What value of x minimizes L^2?

(c) What is the minimum value of L?



49. *Sensitivity to Medicine* (*continuation of Exercise 48, Section 3.3*) Find the amount of medicine to which the body is most sensitive by finding the value of M that maximizes the derivative dR/dM.

50. *Selling Backpacks* It costs you c dollars each to manufacture and distribute backpacks. If the backpacks sell at x dollars each, the number sold is given by

$$n = \frac{a}{x - c} + b(100 - x),$$

where a and b are certain positive constants. What selling price will bring a maximum profit?

Standardized Test Questions

You may use a graphing calculator to solve the following problems.

51. True or False A continuous function on a closed interval must attain a maximum value on that interval. Justify your answer.

52. True or False If $f'(c) = 0$ and $f(c)$ is not a local maximum, then $f(c)$ is a local minimum. Justify your answer.

53. Multiple Choice Two positive numbers have a sum of 60. What is the maximum product of one number times the square of the second number?

(A) 3481

(B) 3600

(C) 27,000

(D) 32,000

(E) 36,000

54. Multiple Choice A continuous function f has domain $[1, 25]$ and range $[3, 30]$. If $f'(x) < 0$ for all x between 1 and 25, what is $f(25)$?

(A) 1

(B) 3

(C) 25

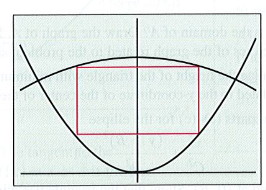
(D) 30

(E) impossible to determine from the information given

55. Multiple Choice What is the maximum area of a right triangle with hypotenuse 10?

(A) 24 **(B)** 25 **(C)** $25\sqrt{2}$ **(D)** 48 **(E)** 50

56. Multiple Choice A rectangle is inscribed between the parabolas $y = 4x^2$ and $y = 30 - x^2$ as shown below:



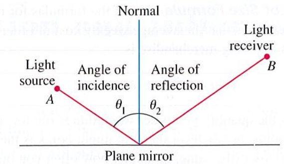
[–3, 3] by [–2, 40]

What is the maximum area of such a rectangle?

(A) $20\sqrt{2}$ **(B)** 40 **(C)** $30\sqrt{2}$ **(D)** 50 **(E)** $40\sqrt{2}$

Explorations

57. *Fermat's Principle in Optics* Fermat's principle in optics states that light always travels from one point to another along a path that minimizes the travel time. Light from a source A is reflected by a plane mirror to a receiver at point B, as shown in the figure. Show that for the light to obey Fermat's principle, the angle of incidence must equal the angle of reflection, both measured from the line normal to the reflecting surface. (This result can also be derived without calculus. There is a purely geometric argument, which you may prefer.)



58. *Tin Pest* When metallic tin is kept below 13.2°C, it slowly becomes brittle and crumbles to a gray powder. Tin objects eventually crumble to this gray powder spontaneously if kept in a cold climate for years. The Europeans who saw tin organ pipes in their churches crumble away years ago called the change *tin pest* because it seemed to be contagious. And indeed it was, for the gray powder is a catalyst for its own formation.

A *catalyst* for a chemical reaction is a substance that controls the rate of reaction without undergoing any permanent change in itself. An *autocatalytic reaction* is one whose product is a catalyst for its own formation. Such a reaction may proceed slowly at first if the amount of catalyst present is small and slowly again at the end, when most of the original substance is used up. But in between, when both the substance and its catalyst product are abundant, the reaction proceeds at a faster pace.

In some cases it is reasonable to assume that the rate $v = dx/dt$ of the reaction is proportional both to the amount of the original substance present and to the amount of product. That is, v may be considered to be a function of x alone, and

$$v = kx(a - x) = kax - kx^2,$$

where

x = the amount of product,

a = the amount of substance at the beginning,

k = a positive constant.

At what value of x does the rate v have a maximum? What is the maximum value of v?

59. *How We Cough* When we cough, the trachea (windpipe) contracts to increase the velocity of the air going out. This raises the question of how much it should contract to maximize the velocity and whether it really contracts that much when we cough.

Under reasonable assumptions about the elasticity of the tracheal wall and about how the air near the wall is slowed by friction, the average flow velocity v (in cm/sec) can be modeled by the equation

$$v = c(r_0 - r)r^2, \quad \frac{r_0}{2} \le r \le r_0,$$

where r_0 is the rest radius of the trachea in cm and c is a positive constant whose value depends in part on the length of the trachea.

(a) Show that v is greatest when $r = (2/3)r_0$, that is, when the trachea is about 33% contracted. The remarkable fact is that X-ray photographs confirm that the trachea contracts about this much during a cough.

(b) Take r_0 to be 0.5 and c to be 1, and graph v over the interval $0 \le r \le 0.5$. Compare what you see to the claim that v is a maximum when $r = (2/3)r_0$.

60. *Wilson Lot Size Formula* One of the formulas for inventory management says that the average weekly cost of ordering, paying for, and holding merchandise is

$$A(q) = \frac{km}{q} + cm + \frac{hq}{2},$$

where q is the quantity you order when things run low (shoes, radios, brooms, or whatever the item might be), k is the cost of placing an order (the same, no matter how often you order), c is the cost of one item (a constant), m is the number of items sold each week (a constant), and h is the weekly holding cost per item (a constant that takes into account things such as space, utilities, insurance, and security).

(a) Your job, as the inventory manager for your store, is to find the quantity that will minimize $A(q)$. What is it? (The formula you get for the answer is called the *Wilson lot size formula.*)

(b) Shipping costs sometimes depend on order size. When they do, it is more realistic to replace k by $k + bq$, the sum of k and a constant multiple of q. What is the most economical quantity to order now?

61. *Production Level* Show that if $r(x) = 6x$ and $c(x) = x^3 - 6x^2 + 15x$ are your revenue and cost functions, then the best you can do is break even (have revenue equal cost).

62. *Production Level* Suppose $c(x) = x^3 - 20x^2 + 20{,}000x$ is the cost of manufacturing x items. Find a production level that will minimize the average cost of making x items.

Extending the Ideas

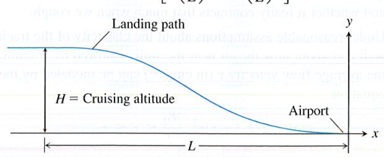
63. *Airplane Landing Path* An airplane is flying at altitude H when it begins its descent to an airport runway that is at horizontal ground distance L from the airplane, as shown in the figure. Assume that the landing path of the airplane is the graph of a cubic polynomial function $y = ax^3 + bx^2 + cx + d$ where $y(-L) = H$ and $y(0) = 0$.

(a) What is dy/dx at $x = 0$?

(b) What is dy/dx at $x = -L$?

(c) Use the values for dy/dx at $x = 0$ and $x = -L$ together with $y(0) = 0$ and $y(-L) = H$ to show that

$$y(x) = H\left[2\left(\frac{x}{L}\right)^3 + 3\left(\frac{x}{L}\right)^2\right].$$



In Exercises 64 and 65, you might find it helpful to use a CAS.

64. *Generalized Cone Problem* A cone of height h and radius r is constructed from a flat, circular disk of radius a in. as described in Exploration 1.

(a) Find a formula for the volume V of the cone in terms of x and a.

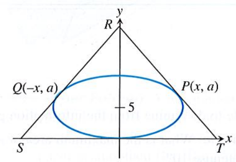
(b) Find r and h in the cone of maximum volume for $a = 4, 5, 6, 8$.

(c) **Writing to Learn** Find a simple relationship between r and h that is independent of a for the cone of maximum volume. Explain how you arrived at your relationship.

65. *Circumscribing an Ellipse* Let $P(x, a)$ and $Q(-x, a)$ be two points on the upper half of the ellipse

$$\frac{x^2}{100} + \frac{(y-5)^2}{25} = 1$$

centered at $(0, 5)$. A triangle RST is formed by using the tangent lines to the ellipse at Q and P as shown in the figure.



(a) Show that the area of the triangle is

$$A(x) = -f'(x)\left[x - \frac{f(x)}{f'(x)}\right]^2,$$

where $y = f(x)$ is the function representing the upper half of the ellipse.

(b) What is the domain of A? Draw the graph of A. How are the asymptotes of the graph related to the problem situation?

(c) Determine the height of the triangle with minimum area. How is it related to the y-coordinate of the center of the ellipse?

(d) Repeat parts (a)–(c) for the ellipse

$$\frac{x^2}{C^2} + \frac{(y-B)^2}{B^2} = 1$$

centered at $(0, B)$. Show that the triangle has minimum area when its height is $3B$.

5.5 Linearization and Differentials

What you will learn about . . .

- Linear Approximation
- Differentials
- Estimating Change with Differentials
- Absolute, Relative, and Percentage Change
- Sensitivity to Change
- Newton's Method

and why . . .

Engineering and science depend on approximations in most practical applications; it is important to understand how approximation techniques work.

Linear Approximation

In our study of the derivative we have frequently referred to the "tangent line to the curve" at a point. What makes that tangent line so important mathematically is that it provides a *useful representation of the curve itself* if we stay close enough to the point of tangency. We say that differentiable curves are always **locally linear,** a fact that can best be appreciated graphically by zooming in at a point on the curve, as Exploration 1 shows.

EXPLORATION 1 Appreciating Local Linearity

The function $f(x) = (x^2 + 0.0001)^{1/4} + 0.9$ is differentiable at $x = 0$ and hence "locally linear" there. Let us explore the significance of this fact with the help of a graphing calculator.

1. Graph $y = f(x)$ in the "ZoomDecimal" window. What appears to be the behavior of the function at the point $(0, 1)$?
2. Show algebraically that f is differentiable at $x = 0$. What is the equation of the tangent line at $(0, 1)$?
3. Now zoom in repeatedly, keeping the cursor at $(0, 1)$. What is the long-range outcome of repeated zooming?
4. The graph of $y = f(x)$ eventually looks like the graph of a line. What line is it?

We hope that this exploration gives you a new appreciation for the tangent line. As you zoom in on a differentiable function, its graph at that point actually seems to *become* the graph of the tangent line! This observation—that even the most complicated differentiable curve behaves locally like the simplest graph of all, a straight line—is the basis for most of the applications of differential calculus. It is what allows us, for example, to refer to the derivative as the "slope of the curve" or as "the velocity at time t_0."

Algebraically, the principle of local linearity means that the *equation* of the tangent line defines a function that can be used to *approximate* a differentiable function near the point of tangency. In recognition of this fact, we give the equation of the tangent line a new name: the *linearization of f at a*. Recall that the tangent line at $(a, f(a))$ has point-slope equation $y - f(a) = f'(x)(x - a)$ (Figure 5.44).

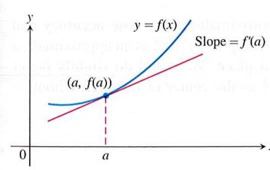


Figure 5.44 The tangent to the curve $y = f(x)$ at $x = a$ is the line $y = f(a) + f'(a)(x - a)$.

DEFINITION Linearization

If f is differentiable at $x = a$, then the equation of the tangent line,

$$L(x) = f(a) + f'(a)(x - a),$$

defines the **linearization of *f* at *a*.** The approximation $f(x) \approx L(x)$ is the **standard linear approximation of *f* at *a*.** The point $x = a$ is the **center** of the approximation.

EXAMPLE 1 Finding a Linearization

Find the linearization of $f(x) = \sqrt{1 + x}$ at $x = 0$, and use it to approximate $\sqrt{1.02}$ without a calculator. Then use a calculator to determine the accuracy of the approximation.

continued